Project Report: Greenhouse Gases in Exoplanet Atmospheres

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English Abstract

The main goal of this project is to estimate the temperature of a planet given its atmospheric composition. Despite making multiple simplifications and assumptions, the model predicts equilibrium temperatures which resemble literature values of planetary temperatures without greenhouse effect. The estimated temperatures of the endoplanets with greenhouse effect tend to be lower than the currently available data for these temperatures. Especially for Venus, the estimated temperature is lower than current measurements suggest. Lastly, the model will be used to find approximate lower and upper bounds to the temperature of Proxima Centauri b, the closest exoplanet to our solar system. We can conclude that the model gives appropriate estimations of the equilibrium temperatures for the investigated endoplanets in the hypothetical case that the greenhouse effect is absent. However, the calculated temperatures do not resemble the final temperatures with greenhouse effect for every planet.

Dutch Abstract

Het doel van dit project is om de temperatuur van een planeet te bepalen gegeven de samenstelling van de atmosfeer. Ondanks dat er veel benaderingen en veronderstellingen worden gemaakt in het model, geeft het voor de temperatuur zonder broeikaseffect toch resultaten die dicht bij de literatuurwaarden liggen. De geschatte waarden voor de temperatuur van de endoplaneten met broeikaseffect in rekening gebracht, liggen over het algemeen lager dan de gemeten data die op dit moment beschikbaar zijn. Vooral voor Venus ligt de geschatte temperatuur lager dan de werkelijke metingen. Ten slotte wordt de temperatuur van Proxima Centauri b, de dichtstbijzijnde exoplaneet, afgeschat. Het model geeft gepaste benaderingen van de equilibrium temperatuur voor de onderzochte endoplaneten in het geval dat er geen greenhouse effect aanwezig zou zijn. De resultaten voor de temperatuur met greenhouse effect wijken echter meer af van de literatuurwaarden.

Contents

1 Introduction

There are several properties that a planet should possess in order for it to be classified as a habitable planet. One of those criteria is the existence of liquid water on the planet's surface. This criterion is highly dependent on the temperature of the planet. Therefore, this temperature is valuable information and it is this planetary temperature that will be discussed in the project.

The habitable zone around a star is usually only calculated using the distance from the planet to its star and the radius and temperature of the star, as will be discussed later on. However, this range of radii does not include any effects of the planet's atmosphere. It is this atmospheric influence that gives rise to the greenhouse effect. The greenhouse effect changes the equilibrium temperature of the planet, such that liquid water could exist outside of the usual habitable zone around a star and therefore, there is a possibility of finding life forms outside of the usual habitable zone.

The goal of this project is to estimate the final temperature of a planet given its atmospheric composition by using a simplified model of the interactions between the atmosphere, the planet and outer space. The main form of the model was derived by D. Petit and S. Kiefer in [1] except for the sections about planetary albedo (section 2.4), eccentricity and radiant flux (section 2.5) and temperature boundaries (section 2.6) which were added to the model by the authors of this paper.

2 Model

2.1 Three system model

In this project, we are assuming the three system model, in which we presume three distinct systems, namely the planet itself (also named ground), the atmosphere and outer space as illustrated in Figure 1. The main corner stone of the model is that it assumes that all systems communicate only through radiation. In Figure 1, the energy per second that is radiated from one system to another is illustrated by the symbol L_{xy} where the first subscript denotes the emitting body and the second subscript stands for the receiving system e.g. L_{ga} should be read as "the energy per second from the ground (g) to the atmosphere (a)".

Furthermore, it is assumed that both the planet and the star are perfect black bodies, which means they absorb all radiation they receive and emit

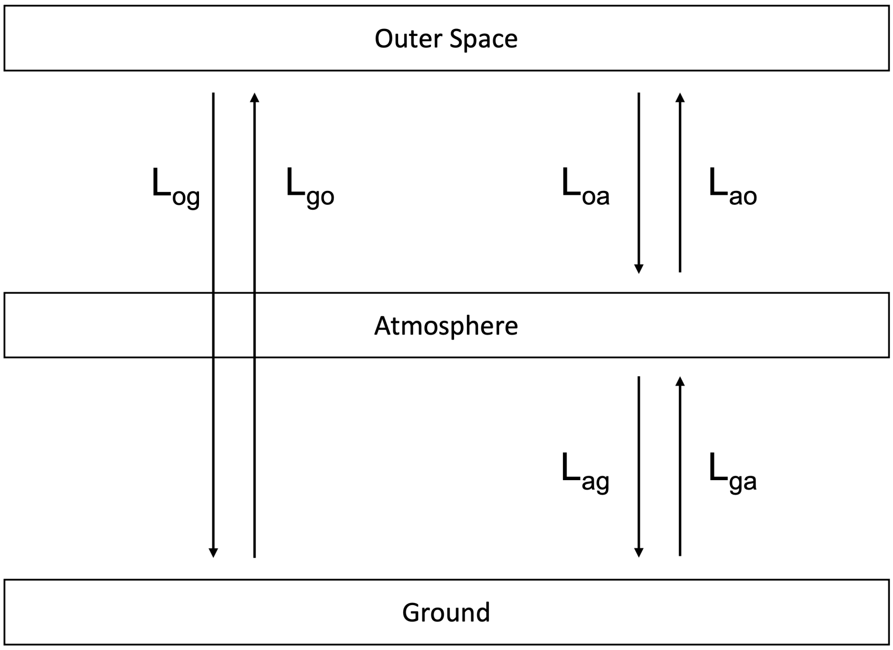

Figure 1: Radiated energy per second of the 3-system model

isotropically according to Planck's formula for black body radiation, this formula is stated as:

$$B_{\mathrm{bb}}(\lambda, T) = \frac{2hc^2}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{\lambda k_B T}\right) - 1}, \tag{1}$$

where λ is the wavelength and T is the temperature of the body.

The atmosphere of the planet is also assumed to emit as a black body. However, for the radiation of the star to be able to reach the ground, the atmosphere does not absorb all incoming radiation and is therefore not absorbing like a black body.

Only a fraction of the total radiance emitted by the star reaches the planet, the effective radiance is given by

$$B_{\text{star, eff}} = \frac{\pi r^2}{4\pi d^2} B_{\text{star}}, \tag{2}$$

with r the radius of the planet and d the distance between the star and the planet.

The total energy that is emitted per second is obtained by integrating the radiance over all wavelengths and over the whole surface of the body:

$$\begin{aligned} L(T) &= \int_A dA \int_0^\infty d\lambda B(\lambda, T), \\ &= 4\pi R^2 \sigma T^4, \end{aligned} \tag{3}$$

where σ is the Boltzmann constant, R is the radius of the emitting body and T is its temperature. In this calculation, it is assumed that both the planet and its star are perfect spheres.

From these assumptions, the equilibrium temperature of the planet, i.e. the temperature without greenhouse effect, can be estimated as

$$T_{\text{eq}} = \left(\frac{R_{star}^2}{4}\right)^{\frac{1}{4}} \cdot \frac{T_{\text{star}}}{\sqrt{d}} \tag{4}$$

This formula does not yet take the greenhouse effect into account, nor the reflectivity or eccentricity of the planet.

2.2 Radiative transfer

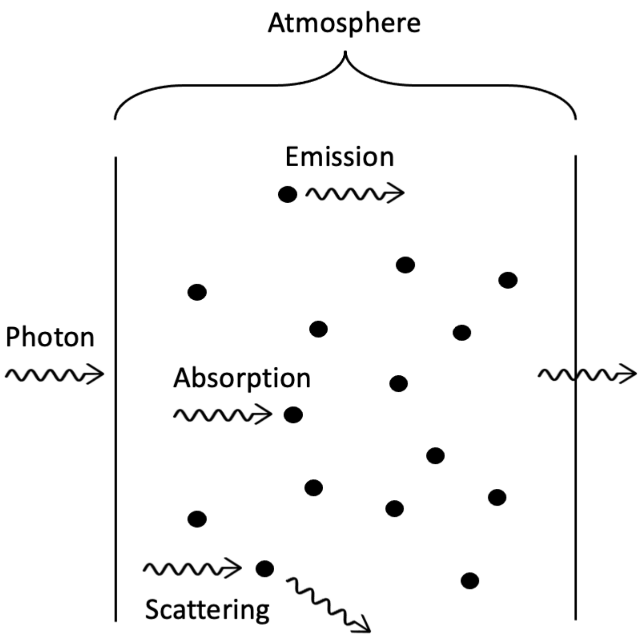

Figure 2: Schematic representation of three processes of radiative transfer.

The atmosphere, acting as a cloud of particles, interacts with radiation coming from the star and from the planet through processes of absorption, emission and scattering. These processes are visually represented in Figure 2. Emission can be due to spontaneous emission of the molecules in the atmosphere or due to re-emission of previously absorbed photons. The incident photons from the star or the planet are either being absorbed or scattered by the atmosphere or they go through it. However, in this model, scattering is neglected and we are only concerned with the interaction due to absorption and emission.

This interaction can be described by the radiative transfer equation:

$$\frac{\mathrm{d}B(\lambda, T, z)}{\mathrm{d}z} = -\alpha(\lambda, z)B(\lambda, T, z) + \beta(\lambda, z) \tag{5}$$

where z donates the path through the atmosphere, α is the height dependent absorption and $\beta(\lambda, z)$ quantifies the height dependent emission of the atmosphere. In this project, it is assumed that emission and absorption are separate processes.

Suppose s is a molecule (also named species) that is present in the atmosphere of a considered planet, then the absorption coefficient α can be written as

$$\alpha(\lambda, z) = \sum_s \sigma_s(\lambda)c_s(z)\rho(z), \tag{6}$$

that is, α is the sum over all species of the effective cross section σ_s of s times the height dependent concentration of s (between 0 and 1) times the height dependent atmospheric density. In the case of absorption, the solution to equation (5) is given by

$$B_{\mathrm{abs}}(\lambda, T, z) = B_{\mathrm{bb}}(\lambda, T) \exp\left(-\sum_s \sigma_s(\lambda)N_s(z)\right), \tag{7}$$

where the column number density N_s for a species s is defined as

$$N_s(z) = \int_{z_0}^{z} c_s(z')\rho(z')dz'. \tag{8}$$

The cross section σ_s in (7) depends on λ (and on T_{eq}). Wien's law states that the most radiated wavelength of a black body is inversely proportional to its equilibrium temperature, thus the star emits more at lower wavelengths than the planet. It is this difference in wavelength and hence also in σ_s that causes the greenhouse effect.

For most gases in the atmosphere, these cross-sections are usually greater in the high wavelength regions than in the low wavelength regions and this causes the average temperature of the planet to increase, giving rise to the positive greenhouse effect. It is also possible that certain species absorb more radiation in low wavelength regions than in high wavelength regions. This could cause a decrease in equilibrium temperature, also called a negative greenhouse effect.

2.2.1 Column number density

The quantity N_s defined in (8) is difficult to use, as it needs the height dependent concentration of each molecule individually. Hence we will assume that the concentration is independent of z. This approximation is acceptable for the species used in our results. However, for other molecules this approximation might not be valid, ozone for instance, is absent close to the ground while present in the stratosphere. For the Earth though, ozone is only the fourth most contributing gas to the greenhouse effect and it is not present in Mars' and Venus' atmosphere; consequently, we will not consider it further on.

Assuming that concentration does not depend on the height allows us to define a species-independent column number density:

$$N = \int_{z_0}^{z} \rho(z')dz', \tag{9}$$

such that equation (7) reduces to

$$B_{\text{abs}}(\lambda, T, z) = B_{\text{bb}}(\lambda, T) \exp\left(-N \sum_s \sigma_s(\lambda) c_s(z)\right). \tag{10}$$

The height dependent density will be modelled by the following formula (taken over from [2]):

$$\rho(z) = \frac{p_0 N_A}{RT_0} \left(1 - \frac{Lz}{T_0}\right)^{\frac{gM}{RL}-1}, \tag{11}$$

where p_0 [Pa] is the pressure at height $z = 0$, $N_A = 6.02 \cdot 10^{23}$ mol^{-1} is Avogadro's constant, $R = 8.314$ J/(mol$\cdot$ K) is the ideal gas constant, T_0 [K] is the ground temperature, L [K/m] is the lapse rate (which quantifies the temperature decrease with increasing height), g is the gravity constant of the planet and M is the molar mass of the surface gas (e.g. dry air for Earth or CO_2 for Mars). Equation (11) assumes that the temperature decreases linearly as z increases, which is an acceptable approximation for low heights. Putting this equation in (9) gives

$$N = \frac{p_0 N_A}{Mg} \left(1 - \left(1 - \frac{Lz}{T_0} \right)^{\frac{gM}{RL}} \right). \tag{12}$$

The dimension of N is given as m^{-2}, this implies that N can be viewed as a measure for the amount of molecules per unit surface with which an incoming photon could interact.

2.3 Derivations of the greenhouse factor

In this model we make the assumption that all systems are in thermal equilibrium, therefore the incoming energy has to be equal to the outgoing energy, giving rise to the following equations.

$$\begin{aligned} L_{ga} + L_{go} &= L_{ag} + L_{og}, \\ L_{ag} + L_{ao} &= L_{ga} + L_{oa}, \\ L_{og} + L_{oa} &= L_{go} + L_{ao}. \end{aligned}$$

The use of fractions as opposed to total energy values will prove useful later on. Hence we define the following quantities:

$$\begin{aligned} f_{oa} &= \frac{L_{oa}}{L_{oa} + L_{og}}, \\ f_{ga} &= \frac{L_{ga}}{L_{ga} + L_{go}}, \\ g^* &= \frac{L_{ag}}{L_{ag} + L_{ao}}. \end{aligned}$$

g^* stands for the fraction of energy that the atmosphere emits to the planet relative to the total outgoing energy. Because we assume the atmosphere to emit as a black body, L only depends on the surface area of the boundary layers:

$$g^* = \frac{4\pi r^2}{4\pi \left(r^2 + (r + z_a)^2 \right)} \tag{13}$$

where z_a is the width of the atmosphere. However, z_a is often negligible in comparison with r, such that $g^* \approx 0.5$. In the example of Earth, the width of the atmosphere that gives a significant contribution to the greenhouse effect is about 10 km, while $r = 6371$ km, hence $g^* = 0.4992$ and so the approximation of 0.5 is valid. Nevertheless, exoplanets with very high densities could attract lighter molecules and hence have a thicker atmosphere, while their radius is relatively small, such that expression (13) should be used instead of 0.5.

The energy radiated by the planet L_{planet} $(= L_{go} + L_{ga})$ relates to the one radiated by the outer space L_{space} $(= L_{oa} + L_{og})$:

$$L_{\text{planet}} = \frac{1 - (1 - g^*) f_{oa}}{1 - g^* f_{ga}} L_{\text{space}}$$

and the greenhouse factor is defined as

$$G_H = \frac{L_{\text{planet}}}{L_{\text{space}}} = \frac{1 - (1 - g^*) f_{oa}}{1 - g^* f_{ga}}. \tag{14}$$

The two fractions f_{oa} and f_{ga} are related to the absorbed radiation:

$$\begin{aligned} f_{oa} &= 1 - \frac{\int_0^\infty B_{\text{abs}}(\lambda, T_{\text{star}}, z) d\lambda}{\int_0^\infty B_{\text{bb}}(\lambda, T_{\text{star}}) d\lambda} \\ f_{ga} &= 1 - \frac{\int_0^\infty B_{\text{abs}}(\lambda, T_{\text{planet}}, z) d\lambda}{\int_0^\infty B_{\text{bb}}(\lambda, T_{\text{planet}}) d\lambda} \end{aligned} \tag{15}$$

The resulting final temperature for the planet becomes

$$T = \left(\frac{R_{\text{star}}^2}{4} \right)^{\frac{1}{4}} \cdot \frac{T_{\text{star}}}{\sqrt{d}} \cdot (G_H)^{\frac{1}{4}}, \tag{16}$$

where the greenhouse factor accounts for a correction to T compared to T_{eq} in equation (4).

2.4 Planetary albedo

The previously derived formulas for the temperatures assume that all the incoming light is either absorbed by the atmosphere or passes freely through the atmosphere. However, a fraction of the incoming radiation gets reflected by the molecules in the atmosphere without contributing to the greenhouse effect, this fraction is called the albedo. This approach is equivalent with allowing one single scattering event to occur within the atmosphere and is therefore an adjustment to the original model, where scattering as a whole was neglected.

Generally, the albedo is dependent on the wavelength of the incoming radiation. It can be included in the calculations by multiplying the radiation that is received by the planet with $(1 - A(\lambda))$ where $A(\lambda)$ denotes the wavelength dependent albedo.

The equilibrium temperature of the planet without greenhouse effect then becomes

$$\begin{aligned} T_{\text{eq}} &= \left(\frac{1}{4\pi R_{\text{planet}}^2 \sigma} \int_A \int_0^\infty (1 - A(\lambda) \frac{R_{\text{planet}}^2}{4d^2} B_{\text{star}}(\lambda, T) d\lambda \right)^{\frac{1}{4}} \\ &= \left(\frac{R_{\text{star}}^2 \cdot T_{\text{star}}^4}{4d^2} - \frac{R_{\text{star}}^2}{4\sigma \cdot d^2} \cdot \int_0^\infty A(\lambda) B_{\text{star}}(\lambda, T) d\lambda \right)^{\frac{1}{4}} \end{aligned} \tag{17}$$

The greenhouse factor changes as well, formula (14) is still valid but the expression for f_{oa} becomes

$$\begin{aligned} f_{oa} &= 1 - \frac{L_{og}}{L_{\text{space}}} \\ &= 1 - \frac{\int_0^\infty (1 - A(\lambda)) B_{\text{abs}}(\lambda, T_{\text{star}}, z) d\lambda}{\int_0^\infty (1 - A(\lambda)) B_{\text{bb}}(\lambda, T_{\text{star}}) d\lambda}, \end{aligned} \tag{18}$$

where $B_{\text{bb}}(\lambda, T_{\text{star}})$ is the radiation emitted by the star according to formula (1) and $B_{\text{abs}}(\lambda, T_{\text{star}}, z)$ is this same radiation but multiplied by the absorption factor as given in equation (7).

The main disadvantage of these formulas is that the spectral albedo, that is the albedo in function of wavelength, is not generally known for most planets. However, it is usually possible to find average values of a planet's bond albedo. This is the total fraction of power that is reflected by the planet. These values can be used to approximate the spectral albedo as a constant value, thus deriving the simplified formula

$$T_{\text{eq}} = \left((1 - A) \cdot \frac{R_{\text{star}}^2}{4} \right)^{\frac{1}{4}} \cdot \frac{T_{\text{star}}}{\sqrt{d}} \tag{19}$$

for the equilibrium temperature without greenhouse effect. Additionally, the terms with the albedo in the expression for f_{oa} will cancel out, leaving the greenhouse factor unchanged relative to the calculations without albedo.

2.5 Eccentricity and radiant flux

In equation (4), while deriving the equilibrium temperature without taking the atmosphere into account, it is assumed that the distance from the planet to the star is constant in time. For planets like Earth and Venus this assumption is justified because the eccentricity of those planets is very close to zero.

However, for several exoplanets, the eccentricity of the orbit differs significantly from zero and consequently, d will not be constant in time. In what follows, we will derive the time averages of $L_{\text{star-to-planet}}$, $\frac{1}{d^2}$ and the temperature of ground of the planet without atmosphere, T_{pl} over a full period τ given a certain eccentricity e. The calculations will result in a correction factor for the time average of T_{pl}. We will refer to equations in the textbook "Classical Mechanics" by Gregory ([3]), in which $d = r$.

The time average over one period of the radiant flux emitted by the star and received by the planet is given by:

$$\langle L_{\text{star-to-planet}} \rangle_t = \left\langle \frac{1}{d^2} \right\rangle \pi \sigma {R_{\text{pl}}}^2 {R_{\text{st}}}^2 {T_{\text{st}}}^4, \tag{20}$$

since only d is dependent on time for the time scale of one orbit.

$$\left\langle \frac{1}{d^2} \right\rangle_t = \frac{1}{\tau} \int_0^\tau \frac{1}{d^2} dt, \tag{21}$$

where τ is the period of the orbit. Using the radial motion equation (Gregory p.159 [3]) we get:

$$\frac{d\theta}{L} = \frac{dt}{d^2}, \tag{22}$$

where L is the angular momentum. Inserting (22) in (21), we obtain:

$$\begin{aligned} \left\langle \frac{1}{d^2} \right\rangle_t &= \frac{1}{\tau} \int_0^{2\pi} \frac{d\theta}{L}, \\ &= \frac{1}{\tau} \frac{2\pi}{L}. \end{aligned} \tag{23}$$

The period is given by (Gregory p.174 [3]):

$$\tau = 2\pi \left(\frac{a^3}{\gamma} \right)^{\frac{1}{2}}, \tag{24}$$

where a is the semi-major axis and $\gamma = M_{\text{st}} G$, with M_{st} the mass of the star and G the gravitational constant.

The angular momentum is given by (Gregory p.171 [3]):

$$L^2 = \frac{\gamma b^2}{a}, \tag{25}$$

where b is the semi-minor axis, therefore

$$\frac{1}{L} = \frac{a^{\frac{1}{2}}}{\gamma^{\frac{1}{2}} b}. \tag{26}$$

Inserting (24) and (26) in (23) we obtain:

$$\frac{1}{\tau}\frac{2\pi}{L} = \frac{1}{ab}, \tag{27}$$

which can be rewritten in terms of the eccentricity of the orbit e defined by:

$$e^2 = 1 - \frac{b^2}{a^2}, \tag{28}$$

hence

$$\frac{1}{\tau}\frac{2\pi}{L} = \frac{1}{a^2\sqrt{1-e^2}} = \left\langle \frac{1}{d^2} \right\rangle_t. \tag{29}$$

This result for $\langle d^2 \rangle_{\mathrm{t}}$ was also achieved by Berger, Loutre [4]. It follows that

$$\langle L_{\text{star-to-planet}} \rangle_t = \frac{1}{a^2\sqrt{1-e^2}} \pi\sigma R_{\mathrm{pl}}^2 R_{\mathrm{st}}^2 T_{\mathrm{st}}^4, \tag{30}$$

i.e. given a constant a, the average stellar flux rises with increasing e as $\frac{1}{\sqrt{1-e^2}}$. For $e \neq 0$, the temperature of the planet T_{pl} is not constant in time, thus we will derive the time average of T_{pl} over one orbit. If we assume that after a time $t = n\tau$ with $n \to \infty$ the incoming stellar radiation is approximately equal to the outgoing planetary radiation:

$$\langle L_{\text{star-to-planet}} \rangle_t \approx \langle L_{\text{planet}} \rangle_t, \tag{31}$$

it follows that

$$\frac{1}{a^2\sqrt{1-e^2}} R_{\mathrm{st}}^2 T_{\mathrm{st}}^4 \approx 4\langle T_{\mathrm{pl}} \rangle_t, \tag{32}$$

where $\langle T_{\mathrm{pl}} \rangle_t$ is the time average of the temperature over one period, hence

$$\left(\frac{1}{a^2\sqrt{1-e^2}}\right)^{\frac{1}{4}} \sqrt{\frac{R_{\mathrm{st}}}{2}} T_{\mathrm{st}} \approx \langle T_{\mathrm{pl}} \rangle_t. \tag{33}$$

If $e = 0$, then $a = d$ and more generally the geometrical mean of d is equal to a. If one erroneously inserts a for d in equation (4) and compare with the previous result (33), where the time average is taken, one can see that the result differs by the following correction factor:

$$C_e \equiv \left(\frac{1}{1-e^2}\right)^{\frac{1}{8}}. \tag{34}$$

This factor gives a correction for $\langle T_{\mathrm{pl}}\rangle_t$, where $C_e \geq 0$ and equality holds when $e = 0$. Therefore, we can replace T_{eq} by $\langle T_{\mathrm{pl}}\rangle_t$ in the model, this will be treated further in the discussion.

We can combine this result with the simplified formula for the albedo (19) to obtain:

$$T_{\mathrm{eq}} = \langle T_{\mathrm{pl}}\rangle_t = \left((1-A)\frac{R_{\mathrm{star}}^2}{4}\right)^{\frac{1}{4}} \frac{T_{\mathrm{star}}}{\sqrt{a}} \left(\frac{1}{1-e^2}\right)^{\frac{1}{8}}. \tag{35}$$

2.6 Temperature boundaries

The model requires a lot of input such as pressure, lapse rate, density distribution, gravity constant, etc. before it can be used to quantify the greenhouse effect on a planet. However, most of this data is solely used to calculate the column number density N. Suppose a planet has an equilibrium temperature close to or in the habitable range. Then N can be regarded as a variable and the temperature with greenhouse effect can be plotted as a function of N for diverse atmospheric compositions. If the graph shows a global maximum or minimum for T and if the planet could still not be habitable in this whole temperature range, then the planet can be excluded from further analysis. By calculating a range of temperatures instead of one value, we can significantly reduce the amount of input data.

There is a priori no certainty that an extremum will occur, but this will turn out to be the case in the results. The process described above will be used to find an upper bound (for four possible atmospheric compositions) for the temperature of Proxima Centauri b (PCb), the closest exoplanet to our solar system. The basic characteristics of this planet are known, but its atmospheric composition and other characteristics like pressure are less certain. This makes from PCb an ideal candidate for the described process. The considered atmospheric compositions are: 100% CO_2, 100% H_20, 50% CO_2-50% H_2O and 33% CO_2-33% H_2O-33% CH_4.

3 Results

3.1 Endoplanets

Our first purpose is to test the model for known planets such as Earth, Mars and Venus. Earth contains multiple greenhouse gases, but the most important (and the ones that will be used in the calculations) are water vapor, carbon dioxide and methane (with corresponding concentrations of 0.0416%, 1.5% and 0.000187%). For Mars and Venus, only carbon dioxide will be used since the atmosphere of these planets is composed of 94.9% and 96.5% of this molecule, respectively. Our model needs data about the planets as input; this data is given in Table 2 in the Appendix. The wavelength dependent cross section $\sigma_s(\lambda)$ of each molecule was taken from the database ExoMol (xsec, see [5])[1]. This data is generated from a line list, hence it has no error bars. Calculated temperatures and column number densities are given in Table 1. The eccentricity and average albedo are already included in T_{eq}, the only difference between T_{eq} and T is the greenhouse factor correction.

Table 1: Calculated T_{eq}, N and T for Earth, Mars and Venus. All given errors are 1σ, standard deviations, calculated with Gaussian error propagation (hence errors from the assumptions made in the model are not included).

Planet	T_{eq} [K]	N [1/m^2]	T [K]
Earth	255.22 ± 0.13	$(1.6899 \pm 0.0003) \cdot 10^{29}$	266.24 ± 0.14
Mars	216.07 ± 0.09	$(2.3949 \pm 0.0019) \cdot 10^{27}$	222.83 ± 0.10
Venus	231.5 ± 0.3	$(1.419 \pm 0.002) \cdot 10^{31}$	246.5 ± 0.4

3.2 Proxima Centauri b

Basic information about Proxima Centauri b (PCb) is needed in order to get an upper bound to its temperature; this data is provided in Table 2 in the Appendix. The resulting equilibrium temperature is

$$T_{\text{eq, PCb}} = 241.2 \pm 0.7 \text{ K},$$

which can be used as equilibrium temperature to find the greenhouse factor and hence the final temperature as a function of the column number density N. Figure 3 contains the different plots for four possible atmospheric compositions.

[1]Used cross sections are of the first isotope of each molecule from wavenumber 100 1/cm to 11900 1/cm (19900 1/cm for Mars and Venus) with space $\Delta\nu = 1$ and equilibrium temperature given in Table 1. Wavenumbers were transformed into wavelengths.

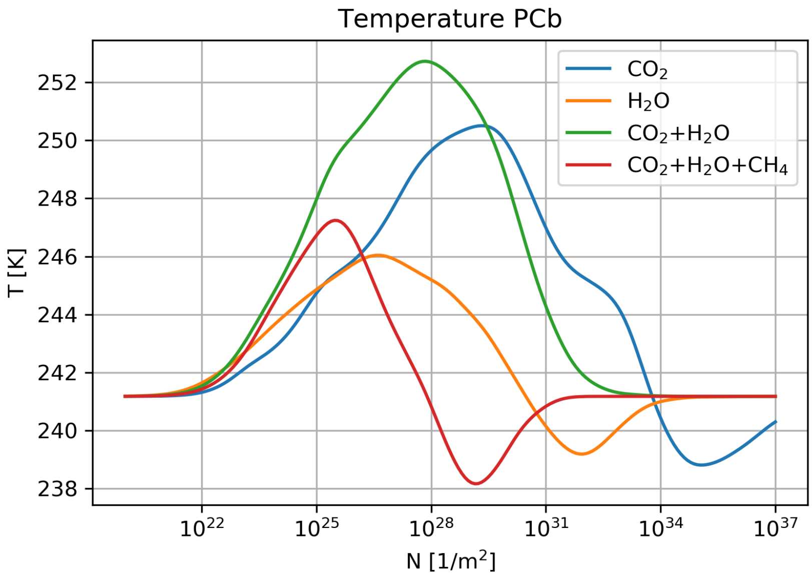

Figure 3: Temperature of PCb as a function of N for four possible atmospheric compositions (blue: 100% CO_2, orange: 100% H_2O, green: 50% CO_2 and 50% H_2O, red: 33% CO_2, 33% H_2O and 33% CH_4).

4 Discussion

4.1 Causes of errors

The model as described in section 2 contains a lot of simplifications. It assumes an isotropic atmosphere with a constant temperature in time. Furthermore the formulas for radiative transfer were strongly simplified since scattering is not included, except for the albedo as a separate phenomenon in a simplified form. In addition to this, emission and absorption are seen as separate processes. Column number densities N_s were also simplified by considering a concentration as constant over height and by using equation (11) as the model for the density.

Another simplification comes from the fact that both the star and the planet are assumed to be perfect black bodies. But it can be observed that the solar spectrum indeed closely resembles the spectrum of a black body and therefore, it is assumed that corrections due to the star not being a perfect black body are rather small.

Effects that might come from other planets around the same star are also excluded. However, these effects are assumed to be very small since planets are cooler and therefore, according to equation (3), emit much less radiation. Eclipses in which a planet blocks part of the solar emission to another planet are rare and therefore do not effect the average equilibrium temperature.

Additionally, the model for the albedo was simplified in the way that an average value was taken, which is independent of the wavelength. In some cases the albedo can also be dependent of the location where the radiation interacts with the planet. This is a consequence of differences in the reflectivity of the planet's surface. The model concerns the total albedo of the planet and surface and does not make distinctions between photons being reflected at the top of the atmosphere or by the planet's surface, as long as the photon is not absorbed by any of them. Therefore, differences in surface area can influence the albedo of a planet. However, since the model already assumes the planet to be in thermal equilibrium, the location where the radiation is mostly absorbed is not relevant and is therefore omitted here. Since most planets are spinning, a different side of the planet will be faced towards the star and this can cause a time dependent total bond albedo. This effect is not incorporated in the model, neither is the effect of changing cloud covers on the albedo.

Another possible cause of errors is the assumption of total equilibrium. Most planets are sending out radiation from their core, this is known as core heating. It means that the total power that is radiated by the planet will be greater than the power it receives from its star. That will cause the predicted temperature for the atmosphere to be lower than the real temperature. However, this error should already be visible in the values for the equilibrium temperature without greenhouse effect and they seem to correspond well with literature values for most planet. Therefore, we assume the effects of core heating to be rather small.

4.2 Endoplanets

4.2.1 Hypothesis tests

For all following hypothesis tests, we define the null hypothesis as; the calculated temperature corresponds with the literature value for the corresponding temperature. In all cases, we used a two-sided-test with significance levels α of 1% and 5%. The measured values are given in Table 1. The literature values that were used can be found in Table 4 in the Appendix.

For the equilibrium temperature without greenhouse effect for Earth, the calculated p-value is 1.98%, hence, the null hypothesis can be accepted at a level of 1% but not at a level of 5%. For the temperature with greenhouse effect for Earth, the p-value is 0.00%. This means that the null hypothesis needs to be rejected on both levels.

The p-value for Mars is 2.62% in the case without greenhouse effect. The null hypothesis can be accepted on a level of 1% but not on a level of 5%. In the case with greenhouse effect, the p-value is 72.57% and the null hypothesis can be accepted on both levels.

For Venus without greenhouse effect, the p-value is 0.36% and in the case with greenhouse effect, the p-value is 0.00%. This means that the null hypothesis needs to be rejected in all cases.

It has to be taken into account that the errors on the calculated temperatures are underestimated because the errors due to the simplifications of the model were not quantified. Therefore, the rejection of some of the results might not be justifiable. Taking this into account, we conclude that the values for the temperatures without greenhouse effect give acceptable estimations for the considered endoplanets but the temperatures with greenhouse effect do not, except for Mars.

4.2.2 Earth and Mars

The equilibrium temperatures for Earth and Mars match with the literature values ([6] and [7]). This was expected since few assumptions are made to calculate these results and the greenhouse effect was not incorporated yet.

Our model predicts an increase of the temperature on Earth of about 11 K, i.e. less than the 32 K one would find in the literature ([6]). This difference could be caused by the simplifications of our model. The latest models of greenhouse effect take other effects into account, like human activity, cross sections of clouds[2] or differences in temperature in our atmosphere.

Mars does not largely increase in temperature because of greenhouse effect (only 6 K). This was expected since the atmosphere of Mars is not dense ([8]), hence photons interact way less with the atmosphere and the greenhouse effect is limited in comparison with Earth and Venus.

4.2.3 Venus

Venus has an equilibrium temperature that is lower than Earth. Even though it is closer to the sun, the dense carbondioxide layer and the sulfuric clouds

[2]Water vapor is included in the model, but clouds are not (only through the average albedo). Clouds also cause greenhouse effect ([9]), but the time and place dependency makes it difficult to quantify.

layer in its atmosphere prohibit the absorption of radiation emitted by the sun. Hence, the albedo is high and, without greenhouse effect, Venus is expected to be colder than Earth. However, the greenhouse effect should increase the temperature of the planet by about 510 K ([7]), but our model does not predict such an increase. The reason for this is not clear, e.g. the internal core could heat the planet, other molecules (other than CO_2) in small concentrations may cause an increase in greenhouse effect or the assumptions of the model may not be valid for Venus.

An internal core that significantly heats the planet is not expected as this temperature increase should only be caused by greenhouse effects ([7]) and CO_2 should be the only important greenhouse gas on Venus. There were multiple assumptions that were made in the model, particularly the model for the column number density could be improved. When comparing Venus to Mars, we find that the most important difference in the greenhouse effect is the much more dense atmosphere of Venus. This causes the difference in column number density. We already found with equation (12) that N is roughly a thousand times greater for Venus than for Mars, but it may be even greater (or lower) if the model used to derive equation (12) is not suitable for Venus.

In the same way as for PCb, we can plot T as a function of N to verify that N is not the cause of the underestimation of the greenhouse factor (see Figure 4). For any acceptable value of N, the temperature of Venus can not be concluded higher than 250 K by the model, thus the false prediction cannot be explained by a wrong column number density.

Another possibility is that Venus may have had water vapor in its atmosphere millennia ago ([10]), heating the planet up in a faster way than CO_2. When water vapor escaped, carbondioxide just kept the warmth on the planet, but it did not heat it by itself for about 510 K. However, our model does not predict such a huge increase, even if Venus' atmosphere were to be composed of 100% H_2O (see Figure 5 in the Appendix).

A plausible explanation is the runaway greenhouse effect ([11]), i.e. CO_2 could not heat Venus up for 510 K in a short time lapse, but it is possible that a long evolution led to this situation. In other words, our assumption that thermal equilibrium is reached, is not legitimate.

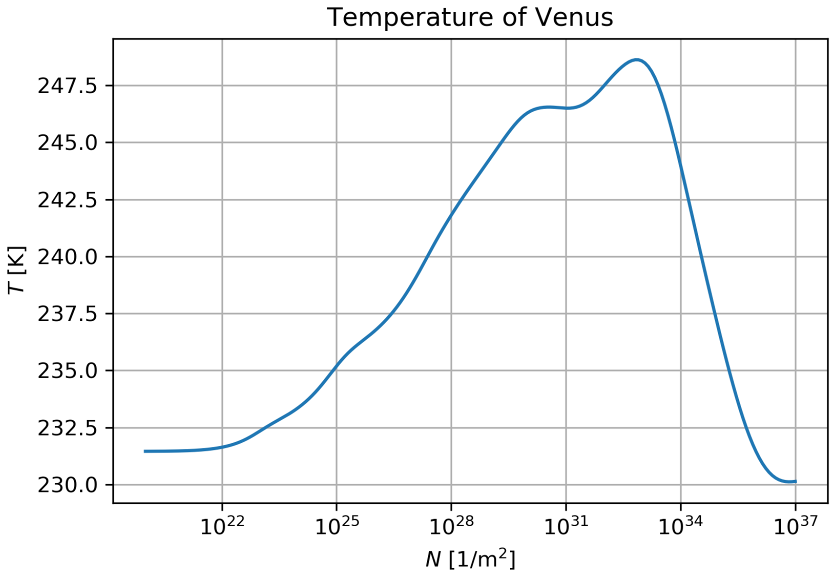

Figure 4: Temperature of Venus as a function of N.

4.3 Proxima Centauri b

Proxima Centauri b (PCb) has an equilibrium temperature that is comparable with that of Earth; PCb is much closer to its star, but it's also smaller than our Sun.

Whatever the considered atmospheric composition, T as a function of N converges to the equilibrium temperature for N very large or very small. This can be explained physically: a low N means less possible interaction and hence less greenhouse effect, while for large N, the difference between f_{oa} and f_{ga} becomes less significant. Hence, the existence of a maximum or minimum looks plausible in a physical context. It seems that PCb could difficultly reach temperatures higher than 251K or lower than 238K; that is still possibly in the habitable range (regardless of other factors than temperature). Of course, only few possible compositions are examined here, such that the temperature could also be out of this range. But PCb has similar properties to Earth, hence, we expect it to contain similar molecules ([12]). A combination of CO_2 and H_2O seems to cause more greenhouse effect, but it is possible that other compositions than the ones that are already considered, reach even higher temperatures.

4.4 Eccentricity and time dependence

The correction factor $C_e = \left(\frac{1}{1-e^2}\right)^{\frac{1}{8}}$ will only have a significant effect if the eccentricity is large enough. For the planets Earth, Mars, Venus and even for Proxima Centauri b, the correction factor is very small. However, numerous exoplanets that have been discovered have an eccentricity that is large enough to make $\langle T_{\text{pl}} \rangle_t$ differ significantly from T_{pl} (equation (4)), calculated by taking $d = a$. For example, if $e = 0.4$, than $C_e = \frac{25}{21}$.

4.4.1 Cross sections and planetary freezing

Even if the amount of incoming stellar radiation and outgoing planetary radiation over one period is balanced and $\langle T_{\text{pl}} \rangle_t$ is a good approximation, it could only be used if the error caused by taking an average temperature for the cross sections and B_{bb} (see equation (1)) is small enough. If d depends on time, the temperature of the planet and the atmosphere will depend on time too and consequently also the cross sections and B_{bb}.

In section 2.5, the calculated correction C_e only takes into account the direct effect on the temperature of the increase in stellar flux, given a certain increase in e. However, due to the variation in temperature, the planet can make a transition into a "snowball state" where the planet (partially) freezes, causing the albedo of the planet to increase, which then cools the planet even more. The onset of this effect has been researched by Dressing et al. ([13]).

It follows that there is a necessary condition in order to justify a constant temperature and a time independent model. The temperature of the planet and the albedo should not vary too much over one orbit, such that the cross sections of the species in the atmosphere and the albedo would not vary in a significant way during one period. The stellar flux decreases with d as $\frac{1}{d^2}$. Consequently, for a large e, the stellar flux will differ significantly between perioastron and apoastron. Therefore, it is a priori not guaranteed that the temperature will not vary too much.

4.4.2 Runaway greenhouse effect

An increase in e would cause the average received stellar flux to rise (30) and would also result in a strong variation in stellar flux. On a planet with liquid water an increase in received stellar radiation can cause water evaporation to increase. Consequently, the amount of water vapor, a potent greenhouse gas, would rise in the atmosphere and as a result increase the greenhouse effect. This rise in temperature could result in a positive feedback loop (Williams and pollard, ([14])). An increase in water vapor in the atmosphere increases the greenhouse effect and as a consequence the temperature rises, increasing the amount of water vapor in the atmosphere even more, which would increase the temperature again. This could result in a runaway greenhouse effect, where all the liquid water could evaporate. To describe this process the following sources where used: Leconte, Forget, Charnay, Wordsworth, Pottier ([15]) and William and Pollard ([14]). The rise in temperature can also cause ice and snow to melt, causing the albedo to decrease. Consequently, this accelerates the rising of the temperature even more. Our current time independent model can not be used for planets in which such a runaway greenhouse effect is occurring.

One could investigate the onset of a runaway greenhouse effect in function of e. This was examined by Williams and Pollard [14] for Earth like planets with eccentric orbits using climate models. They predict an upper limit of ≈ 0.7 for preventing a runaway greenhouse effect on "eccentric Earths".

4.4.3 Time-dependent model

The variation of the planet's temperature depends on its cooling rate. This cooling rate is among other factors dependent on the thermal inertia of the planet, the luminosity of the star and the period of the orbit. Given the high heat capacity of liquid water, we expect that a higher amount of liquid water on a planet would result in a higher thermal inertia and hence, less extreme temperature fluctuations.

In order to get a more accurate description of the Greenhouse effect and more generally in order to find the temperature of a planet at a given time, one would have to take into account the time dependence and adjust the model accordingly. Ideally, one would find expressions of the cooling rate of the planet and the dependence of the albedo and the composition of the atmosphere in function of time.

The distance from a planet to a star is given by (Gregory p. 176 [3]):

$$d = a(1 - e\cos(\psi)), \tag{36}$$

where ψ is the eccentric angle (p.175 Gregory [3]), given by Kepler's equation (p.176 Gregory [3]):

$$t = \frac{\tau}{2\pi}\left(\psi - e\sin\psi\right). \tag{37}$$

A time dependent d can be found by solving Kepler's equation numerically for ψ and inserting the solution in (36). One could try and implement equation (36) in a time dependent model.

If one were to use a constant temperature and a time independent model, it would a priori be better to use $\langle T_{\mathrm{pl}} \rangle_t$ instead of T_{pl} (see equation (4)), calculated with $d = a$.

5 Conclusion

Adjusted with extra factors accounting for the bond albedo of the planet and the eccentricity of its orbit, the calculated equilibrium temperatures without greenhouse effect give reasonable approximations for the considered endoplanets compared with the empirically determined values of the temperatures. This indicates that the model might also give good approximations for the equilibrium temperature without greenhouse effect for several exoplanets, provided that the eccentricity would not be too high. However, more planets should be investigated in order to make reliable conclusions.

One might not be able to use a time independent three system model to calculate the temperature with greenhouse effect if the eccentricity of the orbit is to high. For the examined endoplanets, the calculated greenhouse factor results in an increase in temperature. This is in agreement with the literature and this is also expected given that the considered molecules of the atmospheres have higher cross sections in the higher wavelength regions than in the lower wavelength regions. However, the calculated effects are always lower than those in the literature. Hence, we suspect that the simplifications of the model may not be valid for a significant amount of planets.

The equilibrium temperature of Proxima Centauri b can not be excluded from the range of habitable temperatures. Especially because the predicted greenhouse factor was always lower for endoplanets, hence, we might expect the same for PCb and the planet could show temperatures very similar to Earth. However, other vital factors for habitability must also be explored, e.g. PCb is much closer to its sun, therefore it could receive too much deadly radiation, or solar winds might be too strong.

The calculated temperatures with greenhouse effect are lower than the literature values. However, it is our opinion that the model looks promising. It would be interesting to examine further improvements of the model and then use it to investigate other exoplanets.

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A Data tables and figures

Table 2: Values used in model for Earth, Venus, Mars and Proxima Centauri b. Slash means that it is not used in results. All given errors are uniform 100% intervals. If error is equal to zero, then the value is considered as exact. z_a is exact because it is a considered height and not a measurement.

Planet	Earth	Mars
r [km]	6371.00 ± 0.05	3389.50 ± 0.05
R_{sun} [10^6 m]	695.66 ± 0.14	
T_{sun} [K]	5772.0 ± 0.5	
d [AU]	1 ± 0	1.5240 ± 0.0005
A [/]	0.293 ± 0.005	0.160 ± 0.005
e [/]	0.0167 ± 0.0005	0.0935 ± 0.0005
p_0 [hPa]	1013.25 ± 0.05	6.5180 ± 0.0005
g [m/s^2]	9.80665 ± 0	3.72 ± 0.01
L [K/km]	6.50 ± 0.05	2.50 ± 0.05
M [g/mol]	28.96470 ± 0.00005 (Dry Air)	44.00950 ± 0.00005 (CO_2)
z_a [km]	10 ± 0	50 ± 0

Planet	Venus	Proxima Centauri b
r [km]	6051.80 ± 0.05	/ [3]
R_{sun} [10^6 m]	695.66 ± 0.14	107 ± 3
T_{sun} [K]	5772.0 ± 0.5	$(304 \pm 12) \cdot 10$
d [AU]	0.7230 ± 0.0005	0.049 ± 0.005
A [/]	0.750 ± 0.005	0.270 ± 0.005
e [/]	0 ± 0.005	0.18 ± 0.18
p_0 [hPa]	$(92.0 \pm 0.5) \cdot 10^3$	/
g [m/s^2]	8.870 ± 0.005	/
L [K/km]	7.60 ± 0.05	/
M [g/mol]	44.00950 ± 0.00005 (CO_2)	/
z_a [km]	30 ± 0	/

[3] The radius of the planet is only used to calculate the g-factor in equation (13), but for Proxima Centauri b, g was taken to be 0.5.

Table 3: Average concentrations of greenhouse gases for Earth, Mars and Venus. Slash means that molecule is not implemented in the model for that planet. Given error bars are 100% intervals.

Planet	c_{CO_2} [%]	c_{H_2O} [%]	c_{CH_4} [ppm]
Earth	0.04160 ± 0.00005	1.5 ± 1.5	1.870 ± 0.005
Mars	94.90 ± 0.05	/	/
Venus	96.50 ± 0.05	/	/

Table 4: Literature values of the temperature of considered planets with and without greenhouse effects (see [6] and [7]). All given errors are 1σ standard deviations.

Planet	Without greenhouse [K]	With greenhouse [K]
Earth	256.2 ± 0.4[4]	287.2 ± 0.5
Mars	217.2 ± 0.5	223.2 ± 0.5
Venus	233.2 ± 0.5	743.2 ± 0.5

[4] The standard deviation was calculated by taking the mean of the literature values from [6] and [7].

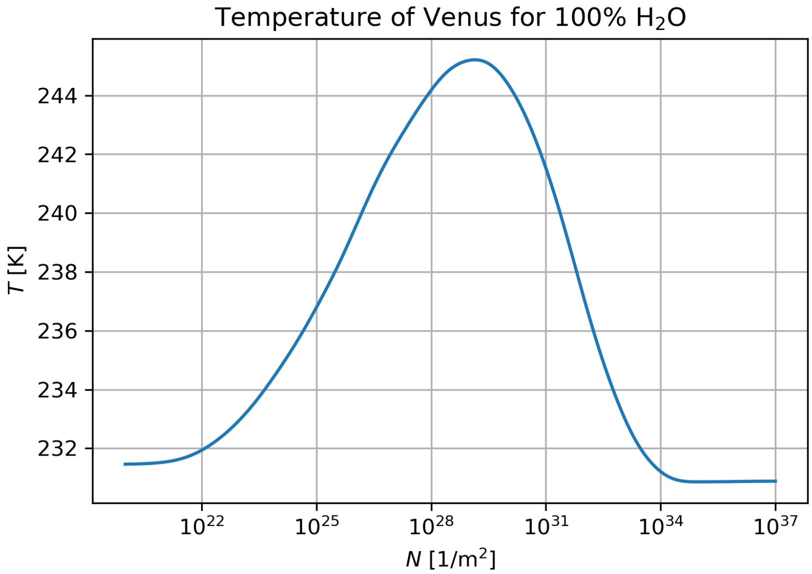

Figure 5: Theoretical temperature of Venus as a function of N if it would be composed of 100% water vapor. T stays below 246 K.